5.1 Midsegment Theorem and Coordinate Proof

Before You used coordinates to show properties of figures.

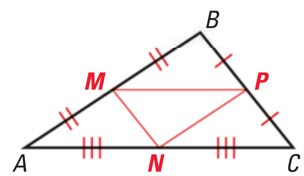
Now You will use properties of midsegments and write coordinate proofs.

Why? So you can use indirect measure to find a height, as in Ex. 35.

Key Vocabulary
- **midsegment of a triangle**
- **coordinate proof**

A **midsegment of a triangle** is a segment that connects the midpoints of two sides of the triangle. Every triangle has three midsegments.

The midsegments of $\triangle ABC$ at the right are $\overline{MP}$, $\overline{MN}$, and $\overline{NP}$.

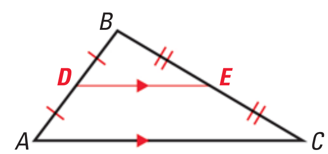


THEOREM *For Your Notebook*

THEOREM 5.1 Midsegment Theorem

The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long as that side.

Proof: Example 5, p. 297; Ex. 41, p. 300

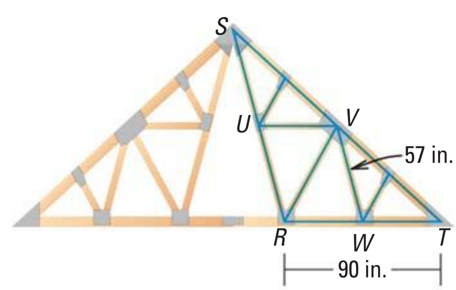


$\overline{DE} \parallel \overline{AC}$ and $DE = \frac{1}{2}AC$

EXAMPLE 1 **Use the Midsegment Theorem to find lengths**

READ DIAGRAMS
In the diagram for Example 1, midsegment $\overline{UV}$ can be called "the midsegment opposite $\overline{RT}$."

CONSTRUCTION Triangles are used for strength in roof trusses. In the diagram, $\overline{UV}$ and $\overline{VW}$ are midsegments of $\triangle RST$. Find UV and RS.



Solution

$$UV = \frac{1}{2} \cdot RT = \frac{1}{2}(90 \text{ in.}) = 45 \text{ in.}$$

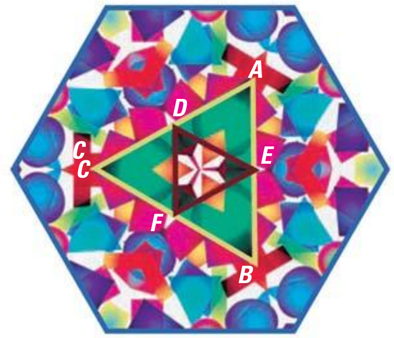
$$RS = 2 \cdot VW = 2(57 \text{ in.}) = 114 \text{ in.}$$

✓ **GUIDED PRACTICE** for Example 1

1. Copy the diagram in Example 1. Draw and name the third midsegment.
2. In Example 1, suppose the distance UW is 81 inches. Find VS.

EXAMPLE 2 Use the Midsegment Theorem

In the kaleidoscope image, $\overline{AE} \cong \overline{BE}$ and $\overline{AD} \cong \overline{CD}$. Show that $\overline{CB} \parallel \overline{DE}$.



Solution

Because $\overline{AE} \cong \overline{BE}$ and $\overline{AD} \cong \overline{CD}$, E is the midpoint of $\overline{AB}$ and D is the midpoint of $\overline{AC}$ by definition. Then $\overline{DE}$ is a midsegment of $\triangle ABC$ by definition and $\overline{CB} \parallel \overline{DE}$ by the Midsegment Theorem.

COORDINATE PROOF A **coordinate proof** involves placing geometric figures in a coordinate plane. When you use variables to represent the coordinates of a figure in a coordinate proof, the results are true for all figures of that type.

EXAMPLE 3 Place a figure in a coordinate plane

Place each figure in a coordinate plane in a way that is convenient for finding side lengths. Assign coordinates to each vertex.

a. A rectangle

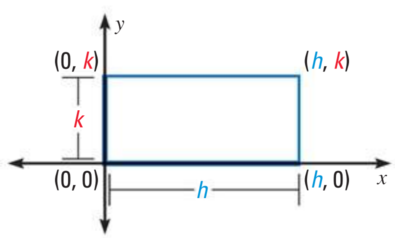
b. A scalene triangle

Solution

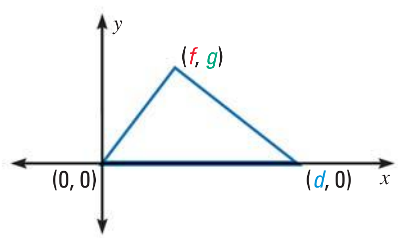
It is easy to find lengths of horizontal and vertical segments and distances from (0, 0), so place one vertex at the origin and one or more sides on an axis.

USE VARIABLES
The rectangle shown represents a general rectangle because the choice of coordinates is based only on the definition of a rectangle. If you use this rectangle to prove a result, the result will be true for all rectangles.

a. Let h represent the length and k represent the width.



b. Notice that you need to use three different variables.



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✓ GUIDED PRACTICE for Examples 2 and 3

3. In Example 2, if F is the midpoint of $\overline{CB}$, what do you know about $\overline{DF}$?

4. Show another way to place the rectangle in part (a) of Example 3 that is convenient for finding side lengths. Assign new coordinates.

5. Is it possible to find any of the side lengths in part (b) of Example 3 without using the Distance Formula? *Explain.*

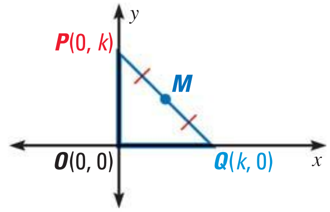
6. A square has vertices (0, 0), $(m, 0)$, and $(0, m)$. Find the fourth vertex.

EXAMPLE 4 Apply variable coordinates

Place an isosceles right triangle in a coordinate plane. Then find the length of the hypotenuse and the coordinates of its midpoint *M*.

ANOTHER WAY
For an alternative method for solving the problem in Example 4, turn to page 302 for the **Problem Solving Workshop**.

Solution



Place $\triangle PQO$ with the right angle at the origin. Let the length of the legs be k. Then the vertices are located at $P(0, k)$, $Q(k, 0)$, and $O(0, 0)$.

Use the Distance Formula to find PQ.

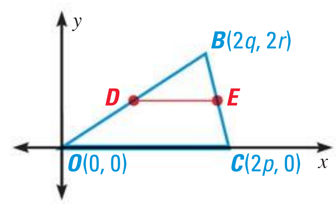
$$PQ = \sqrt{(k - 0)^2 + (0 - k)^2} = \sqrt{k^2 + (-k)^2} = \sqrt{k^2 + k^2} = \sqrt{2k^2} = k\sqrt{2}$$

Use the Midpoint Formula to find the midpoint M of the hypotenuse.

$$M\left(\frac{0 + k}{2}, \frac{k + 0}{2}\right) = M\left(\frac{k}{2}, \frac{k}{2}\right)$$

EXAMPLE 5 Prove the Midsegment Theorem

Write a coordinate proof of the Midsegment Theorem for one midsegment.



GIVEN ▶ $\overline{DE}$ is a midsegment of $\triangle OBC$.

PROVE ▶ $\overline{DE} \parallel \overline{OC}$ and $DE = \frac{1}{2}OC$

Solution

WRITE PROOFS
You can often assign coordinates in several ways, so choose a way that makes computation easier. In Example 5, you can avoid fractions by using $2p$, $2q$, and $2r$.

STEP 1 **Place** $\triangle OBC$ and assign coordinates. Because you are finding midpoints, use $2p$, $2q$, and $2r$. Then find the coordinates of D and E.

$$D\left(\frac{2q + 0}{2}, \frac{2r + 0}{2}\right) = D(q, r) \qquad E\left(\frac{2q + 2p}{2}, \frac{2r + 0}{2}\right) = E(q + p, r)$$

STEP 2 **Prove** $\overline{DE} \parallel \overline{OC}$. The y-coordinates of D and E are the same, so $\overline{DE}$ has a slope of 0. $\overline{OC}$ is on the x-axis, so its slope is 0.

▶ Because their slopes are the same, $\overline{DE} \parallel \overline{OC}$.

STEP 3 **Prove** $DE = \frac{1}{2}OC$. Use the Ruler Postulate to find $\overline{DE}$ and $\overline{OC}$.

$$DE = \left|(q + p) - q\right| = p \qquad OC = \left|2p - 0\right| = 2p$$

▶ So, the length of $\overline{DE}$ is half the length of $\overline{OC}$.

✓ GUIDED PRACTICE for Examples 4 and 5

7. In Example 5, find the coordinates of F, the midpoint of $\overline{OC}$. Then show that $\overline{EF} \parallel \overline{OB}$.

8. Graph the points $O(0, 0)$, $H(m, n)$, and $J(m, 0)$. Is $\triangle OHJ$ a right triangle? Find the side lengths and the coordinates of the midpoint of each side.

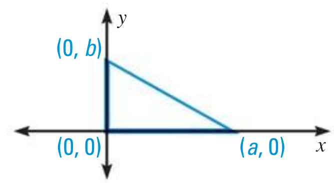
5.1 EXERCISES

HOMEWORK KEY
○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 9, 21, and 37
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 31, and 39

SKILL PRACTICE

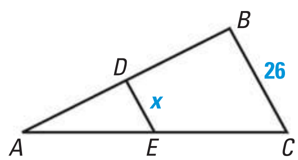
1. **VOCABULARY** Copy and complete: In $\triangle ABC$, D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{AC}$. $\overline{DE}$ is a _?_ of $\triangle ABC$.

2. ★ **WRITING** *Explain* why it is convenient to place a right triangle on the grid as shown when writing a coordinate proof. How might you want to relabel the coordinates of the vertices if the proof involves midpoints?



EXAMPLES 1 and 2 on pp. 295–296 for Exs. 3–11

FINDING LENGTHS $\overline{DE}$ **is a midsegment of** $\triangle ABC$. **Find the value of** x.

3.


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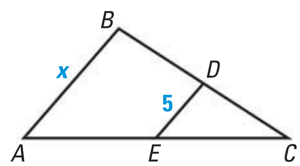
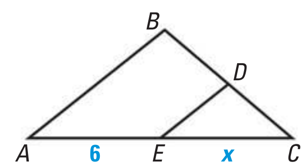
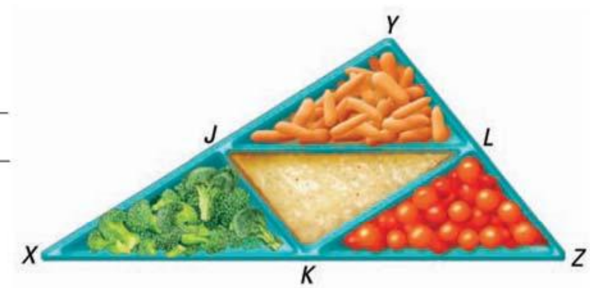


5.


USING THE MIDSEGMENT THEOREM **In** $\triangle XYZ$, $\overline{XJ} \cong \overline{JY}$, $\overline{YL} \cong \overline{LZ}$, **and** $\overline{XK} \cong \overline{KZ}$. **Copy and complete the statement.**

6. $\overline{JK} \parallel$ _?_
7. $\overline{JL} \parallel$ _?_
8. $\overline{XY} \parallel$ _?_
9. $\overline{YJ} \cong$ _?_ $\cong$ _?_
10. $\overline{JL} \cong$ _?_ $\cong$ _?_
11. $\overline{JK} \cong$ _?_ $\cong$ _?_



EXAMPLE 3 on p. 296 for Exs. 12–19

PLACING FIGURES **Place the figure in a coordinate plane in a convenient way. Assign coordinates to each vertex.**

12. Right triangle: leg lengths are 3 units and 2 units
13. Isosceles right triangle: leg length is 7 units
14. Square: side length is 3 units
15. Scalene triangle: one side length is $2m$
16. Rectangle: length is a and width is b
17. Square: side length is s
18. Isosceles right triangle: leg length is p
19. Right triangle: leg lengths are r and s

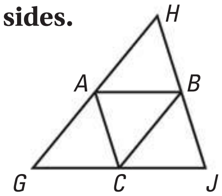
EXAMPLES 4 and 5 on p. 297 for Exs. 20–23

20. **COMPARING METHODS** Find the length of the hypotenuse in Exercise 19. Then place the triangle another way and use the new coordinates to find the length of the hypotenuse. Do you get the same result?

APPLYING VARIABLE COORDINATES **Sketch** $\triangle ABC$. **Find the length and the slope of each side. Then find the coordinates of each midpoint. Is** $\triangle ABC$ **a right triangle? Is it isosceles?** ***Explain.*** **(Assume all variables are positive,** $p \neq q$, **and** $m \neq n$.**)**

21. $A(0, 0)$, $B(p, q)$, $C(2p, 0)$
22. $A(0, 0)$, $B(h, h)$, $C(2h, 0)$
23. $A(0, n)$, $B(m, n)$, $C(m, 0)$

ALGEBRA **Use $\triangle GHJ$, where A, B, and C are midpoints of the sides.**

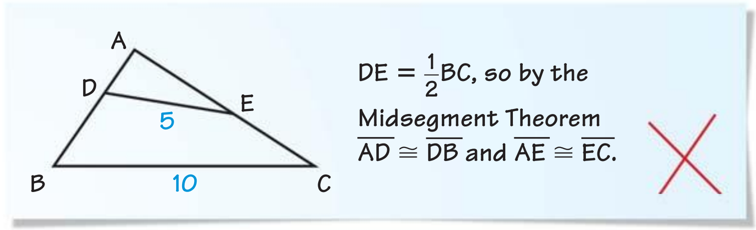


24. If $AB = 3x + 8$ and $GJ = 2x + 24$, what is AB?

25. If $AC = 3y - 5$ and $HJ = 4y + 2$, what is HB?

26. If $GH = 7z - 1$ and $BC = 4z - 3$, what is GH?

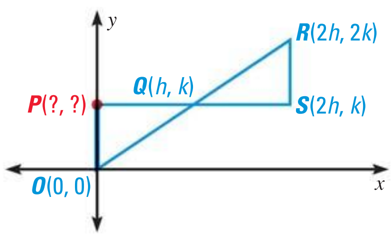
27. **ERROR ANALYSIS** *Explain* why the conclusion is incorrect.



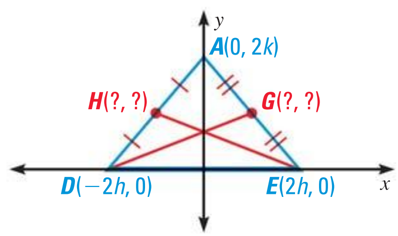
28. **FINDING PERIMETER** The midpoints of the three sides of a triangle are $P(2, 0)$, $Q(7, 12)$, and $R(16, 0)$. Find the length of each midsegment and the perimeter of $\triangle PQR$. Then find the perimeter of the original triangle.

APPLYING VARIABLE COORDINATES **Find the coordinates of the red point(s) in the figure. Then show that the given statement is true.**

29. $\triangle OPQ \cong \triangle RSQ$



30. slope of $\overline{HE} = -$(slope of $\overline{DG}$)



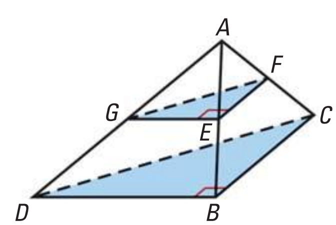
31. ★ **MULTIPLE CHOICE** A rectangle with side lengths $3h$ and k has a vertex at $(-h, k)$. Which point *cannot* be a vertex of the rectangle?

Ⓐ (h, k) Ⓑ $(-h, 0)$ Ⓒ $(2h, 0)$ Ⓓ $(2h, k)$

32. **RECONSTRUCTING A TRIANGLE** The points $T(2, 1)$, $U(4, 5)$, and $V(7, 4)$ are the midpoints of the sides of a triangle. Graph the three midsegments. Then show how to use your graph and the properties of midsegments to draw the original triangle. Give the coordinates of each vertex.

33. **3-D FIGURES** Points A, B, C, and D are the vertices of a *tetrahedron* (a solid bounded by four triangles). $\overline{EF}$ is a midsegment of $\triangle ABC$, $\overline{GE}$ is a midsegment of $\triangle ABD$, and $\overline{FG}$ is a midsegment of $\triangle ACD$.

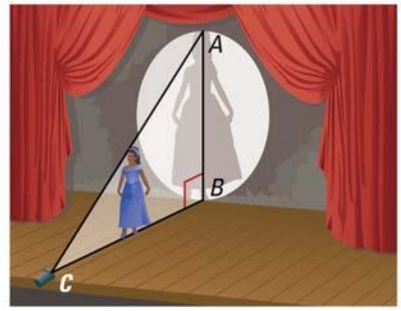
Show that Area of $\triangle EFG = \frac{1}{4} \cdot$ Area of $\triangle BCD$.



34. **CHALLENGE** In $\triangle PQR$, the midpoint of $\overline{PQ}$ is $K(4, 12)$, the midpoint of $\overline{QR}$ is $L(5, 15)$, and the midpoint of $\overline{PR}$ is $M(6.4, 10.8)$. Show how to find the vertices of $\triangle PQR$. *Compare* your work for this exercise with your work for Exercise 32. How were your methods different?

PROBLEM SOLVING

35. **FLOODLIGHTS** A floodlight on the edge of the stage shines upward onto the curtain as shown. Constance is 5 feet tall. She stands halfway between the light and the curtain, and the top of her head is at the midpoint of $\overline{AC}$. The edge of the light just reaches the top of her head. How tall is her shadow?



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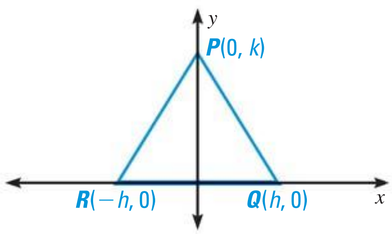
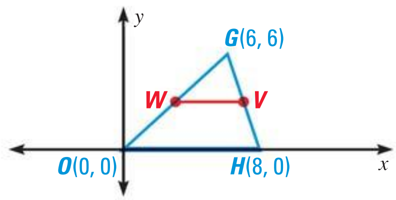
EXAMPLE 5 on p. 297 for Exs. 36–37

COORDINATE PROOF **Write a coordinate proof.**

36. **GIVEN ▶** $P(0, k)$, $Q(h, 0)$, $R(-h, 0)$
PROVE ▶ $\triangle PQR$ is isosceles.



37. **GIVEN ▶** $O(0, 0)$, $G(6, 6)$, $H(8, 0)$, $\overline{WV}$ is a midsegment.
PROVE ▶ $\overline{WV} \parallel \overline{OH}$ and $WV = \frac{1}{2}OH$



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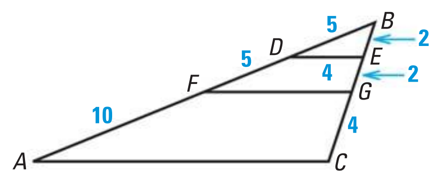
38. **CARPENTRY** In the set of shelves shown, the third shelf, labeled $\overline{CD}$, is closer to the bottom shelf, $\overline{EF}$, than midsegment $\overline{AB}$ is. If $\overline{EF}$ is 8 feet long, is it possible for $\overline{CD}$ to be 3 feet long? 4 feet long? 6 feet long? 8 feet long? *Explain*.

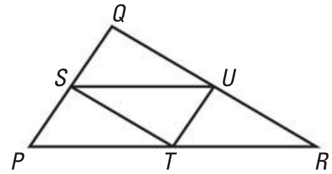
39. ★ **SHORT RESPONSE** Use the information in the diagram at the right. What is the length of side $\overline{AC}$ of $\triangle ABC$? *Explain* your reasoning.



40. **PLANNING FOR PROOF** Copy and complete the plan for proof.

GIVEN ▶ $\overline{ST}$, $\overline{TU}$, and $\overline{SU}$ are midsegments of $\triangle PQR$.
PROVE ▶ $\triangle PST \cong \triangle SQU$

Use __?__ to show that $\overline{PS} \cong \overline{SQ}$. Use __?__ to show that $\angle QSU \cong \angle SPT$. Use __?__ to show that $\angle$ __?__ $\cong \angle$ __?__. Use __?__ to show that $\triangle PST \cong \triangle SQU$.



41. **PROVING THEOREM 5.1** Use the figure in Example 5. Draw the midpoint F of $\overline{OC}$. Prove that $\overline{DF}$ is parallel to $\overline{BC}$ and $DF = \frac{1}{2}BC$.

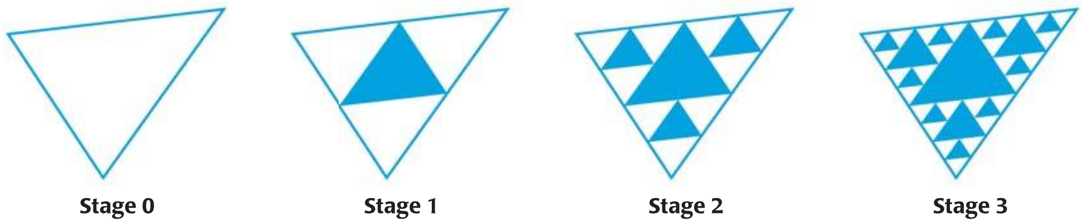
○ = WORKED-OUT SOLUTIONS on p. WS1 ★ = STANDARDIZED TEST PRACTICE

42. **COORDINATE PROOF** Write a coordinate proof.

GIVEN ▶ $\triangle ABD$ is a right triangle, with the right angle at vertex A. Point C is the midpoint of hypotenuse BD.

PROVE ▶ Point C is the same distance from each vertex of $\triangle ABD$.

43. **MULTI-STEP PROBLEM** To create the design below, shade the triangle formed by the three midsegments of a triangle. Then repeat the process for each unshaded triangle. Let the perimeter of the original triangle be 1.



a. What is the perimeter of the triangle that is shaded in Stage 1?

b. What is the total perimeter of all the shaded triangles in Stage 2?

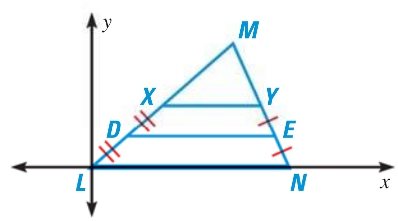
c. What is the total perimeter of all the shaded triangles in Stage 3?

RIGHT ISOSCELES TRIANGLES In Exercises 44 and 45, write a coordinate proof.

44. Any right isosceles triangle can be subdivided into a pair of congruent right isosceles triangles. (*Hint:* Draw the segment from the right angle to the midpoint of the hypotenuse.)

45. Any two congruent right isosceles triangles can be combined to form a single right isosceles triangle.

46. **CHALLENGE** XY is a midsegment of $\triangle LMN$. Suppose $\overline{DE}$ is called a "quarter-segment" of $\triangle LMN$. What do you think an "eighth-segment" would be? Make a conjecture about the properties of a quarter-segment and of an eighth-segment. Use variable coordinates to verify your conjectures.

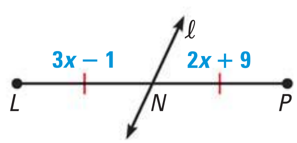


MIXED REVIEW

PREVIEW
Prepare for Lesson 5.2 in Exs. 47–49.

Line ℓ bisects the segment. Find LN. *(p. 15)*

47.



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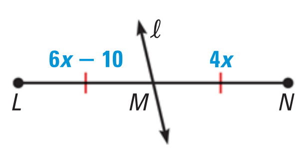

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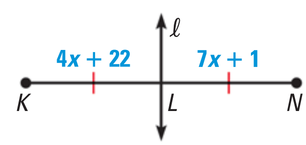
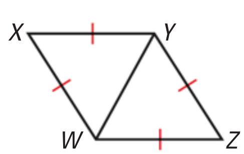
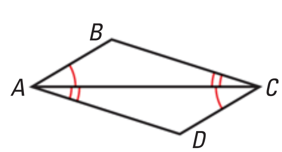


State which postulate or theorem you can use to prove that the triangles are congruent. Then write a congruence statement. *(pp. 225, 249)*

50.



51.



52.